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LETTER TO THE EDITOR

On backbends on percolation backbones

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Abstract. Just above the percolation concentration, a path on the backbone which leads from one side of the lattice to the other is not direct but zigzags through the lattice. Backbends are the portions of the zigzags which go backwards. They are important in the problem of particle transport in strong external fields, as they act as traps and limit the current. The threshold concentration for the proliferation of paths with backbends no longer than a given length L is defined as $p_b(L)$, with the limits $p_b(0) = p_d$ (directed percolation) and $p_b(\infty) = p_c$ (ordinary percolation). The inverse function $\zeta(p)$ is the smallest integer such that, for given p between p_c and p_d, there are paths to infinity on which every backbend is smaller than $\zeta(p)$. This minimal backbend length is computed on a Bethe lattice and shown to diverge as $(p-p_c)^{-1/2}$. It is argued heuristically that on all lattices $\zeta(p)$ is proportional to the correlation length in the limit $p \to p_c$. The chemical lengths of minimal backbend paths on the Bethe lattice are calculated.

Connected paths in a random medium can be characterised in a variety of ways [1]. The nature of connections plays a crucial role in determining transport properties in the medium. For instance, the chemical length [2-8] of shortest paths (measured along the path) between two points is an indicator of the tortuosity of connections, and is presumed to determine signal velocities in a particular physical system [8].

An alternative characterisation we will discuss below is motivated by the problem of particle transport in random media in the presence of a strong external field [9-20]. In this situation, the shortest paths are not necessarily those on which motion is swiftest. This is because typical shortest paths are expected to have arbitrarily long backbends—portions of the path which double back in the direction of the field—which can trap particles for very long times†. Temporally shorter paths—and these are the ones that are eventually effectual in transport if the particles have short-ranged repulsive interactions—require a different characterisation in terms of the longest backbend on the path. This is made clear below.

For definiteness, we model the network by the infinite percolation cluster [21] on a hypercubic lattice from which a fraction $(1-p)$ of sites have been removed at random. Let $\hat{e}$ be a unit vector in the $(1, 1, 1, \ldots)$ direction, normal to two planes D_1 and D_2 with equations $\Sigma_i x_i = \pm \frac{1}{2} d$ respectively. Consider paths—self-avoiding walks (SAW) whose steps lie on the percolation network—that lead from D_1 to D_2 (see figure 1). Along a particular SAW, if $\boldsymbol{R}_n$ is the location of the walk after the nth step, then $r_n \equiv (\boldsymbol{R}_n \cdot \hat{e})$ is the displacement along $\hat{e}$. Let the maximal displacement up to and including step n be

$$M_n = \max_{n' \leq n} (r_{n'}). \tag{1}$$

† We disregard trapping in dead-end branches as here we want to focus only on the properties of the backbone. Branch trapping and its consequences have been studied in references [11-13, 18-20].

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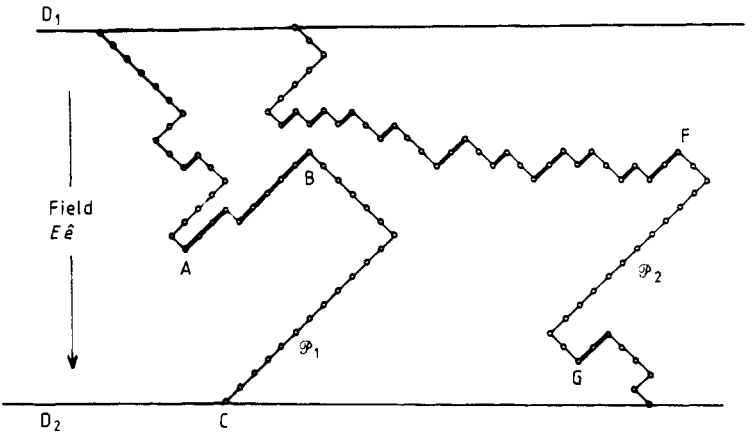

Figure 1. Two paths on the percolation backbend. Backbends are indicated by bold lines. Path $\mathscr{P}_1$ is (chemically) shorter than path $\mathscr{P}_2$. However, in strong enough fields, motion on $\mathscr{P}_2$ is faster than on $\mathscr{P}_1$: it takes much longer to go across the single backbend AB of height eight than the series of shorter backbends of height one or two on path $\mathscr{P}_2$. Movement on segments like BC or PG is in the direction of the field and the time taken between backbends can be neglected compared to backbend traversal times.

The backward displacement (from maximal) is then

$$l_n = M_n - r_n. \tag{2}$$

On each path $\mathscr{P}$ from D_1 to D_2, let $L(\mathscr{P})$ be the maximum value attained by l_n; it is the length of the longest backward excursion, or backbend, on path $\mathscr{P}$. Figure 1 shows two paths $\mathscr{P}_1$ and $\mathscr{P}_2$, on which the value of L is seven and two, respectively.

At fixed p, as the separation of the two planes D_1 and D_2 increases to infinity, let ζ be the *smallest* value of L such that there exist infinitely long paths which connect D_1 and D_2. (Clearly, ζ depends on the fraction p of occupied sites.) In lattice models of the type under discussion, $\zeta(p)$ is the smallest integer such that there are infinitely long paths satisfying the constraint that *every* backbend on the path is *at most* of length $\zeta(p)$.

It is helpful to consider familiar limits. Let p_c and p_d denote the critical probabilities for ordinary [21] and directed [22] percolation. For $p > p_d$, directed paths (which do not have any backbends) proliferate, so that $\zeta(p) = 0$ in the range $p_d < p \leqslant 1$. At the other extreme, for p slightly larger than p_c, connections are very tortuous and involve large backbends. $\zeta(p)$ is large and is expected to diverge as p approaches p_c, as

$$\zeta(p) \sim (p - p_c)^{-\mu}. \tag{3}$$

We will argue that the critical exponent μ is the same as the familiar correlation length exponent ν. However, first we discuss the relevance of backbend length constraints in transport phenomena.

We focus on a particular path $\mathscr{P}_i$ of the network and consider the dynamics of a single particle constrained to move on $\mathscr{P}_i$ in the presence of an external field $E\hat{e}$. The particle moves by nearest-neighbour jumps, with jumps along $\hat{e}$ being more likely than against $\hat{e}$. The field E produces a potential energy which has local minima at the bottoms of backbends (sites A and G in figure 1) and local maxima at their tops (sites B and F). In a backbend of length l, the particle must move against the field and

traverse an energy barrier El. The mean activation time, τ_l, is [20]

$$\tau_l \sim \exp(El/k_B T) \tag{4}$$

where T is the temperature. Denoting the probability of occurrence (per unit length of the path $\mathcal{P}_i$) of a backbend of length l by prob(l), and neglecting the transit time between backbends (e.g. segments BC or FG) which is small as the flow is in the direction of E, an estimate of the mean transit time (per unit length of network) along $\mathcal{P}_i$ is

$$\langle\tau\rangle \sim \sum_l \text{prob}(l) \exp(El/k_B T). \tag{5}$$

The question of whether $\langle\tau\rangle$ diverges or converges for large E is sensitive to estimates of prob(l). If $\mathcal{P}_i$ is a typical shortest path on the percolation network, then prob(l) is expected to decay exponentially for large l. In that case, $\langle\tau\rangle$ diverges if E is large enough.

On the other hand, if the path $\mathcal{P}_i$ is chosen such that *all* backbends on $\mathcal{P}_i$ are smaller than some fixed L, then $\langle\tau\rangle$ is finite. In fact, for large enough fields, $\langle\tau\rangle$ is expected to be smallest (and movement easiest) on paths with the smallest possible value of L, namely $\zeta(p)$. A more detailed account of transport of interacting particles through the full percolation network will be given elsewhere [23].

We now calculate $\zeta(p)$ on a Bethe lattice, which is an endlessly branching structure with no loops. The conventional percolation problem has been solved on this pseudo-lattice [24] and the lengths of shortest paths between distant points have been determined as well [7].

We consider a Bethe lattice with coordination number $2m$ and assign arrows to the bonds of the undiluted lattice. At every site there are m in-pointing and m out-pointing arrows. (Think of an arrow on a bond as representing the component of $\hat{e}$ along that bond.) On an N-stepped SAW leading away from the origin, O, let $s_n = 1(-1)$ if the nth step is taken along (against) the arrow on that bond. Then the displacement in the direction of $\hat{e}$ after n steps is $r_n \equiv (\boldsymbol{R}_n \cdot \hat{e}) = \sum_{k=1}^{n} s_n$ and M_n is the maximum displacement up to that point. We wish to determine the number $Q_N(L)$ of SAW on the undiluted Bethe lattice which start from O and satisfy the constraint

$$l_n \leqslant L \qquad n = 1, 2, \ldots, N. \tag{6}$$

On the face of it, the constraint is non-Markovian since M_n depends on the entire history of the SAW. However, this issue can be sidestepped if we write $Q_N(L)$ in terms of a transfer matrix which keeps track of not only the step variable s_n but also the current backbend length l_n. At the $(n+1)$th step

$$l_{n+1} = l_n - s_{n+1}. \tag{7}$$

There are $m-1$ possible options for the SAW if $s_{n+1} = -s_n$ and m options if $s_{n+1} = s_n$ (except if $l_n = L$ and $s_{n+1} = -1$, in which case the step is not allowed). Explicitly, we have

$$T(s_n, l_n; s_{n+1}, l_{n+1}) = \begin{cases} 0 & \text{if } l_n = L \text{ and } s_{n+1} = -1 \\ [m(m-1)]^{1/2} \exp(K s_n s_{n+1}) \delta_{l_{n+1}, l_n - s_{n+1}} & \text{otherwise} \end{cases} \tag{8}$$

with coupling constant K given by

$$K = \tfrac{1}{2} \ln[m/(m-1)]. \tag{9}$$

The number of SAW consistent with the backbend constraint in equation (6) is then

$$Q_N(L) = \sum_{\{s_n, l_n\}} \prod_{n=1}^{N-1} T(s_n, l_n; s_{n+1}, l_{n+1}) \tag{10}$$

and as $N \to \infty$

$$Q_N(L) \sim [\lambda_0(L)]^N \tag{11}$$

where $\lambda_0(L)$ is the largest eigenvalue of T.

On a randomly diluted Bethe lattice, from which the fraction $(1-p)$ of the sites has been removed, the probability that the endpoint of an N-step SAW is connected to O is p^{N+1}. Since on a Bethe lattice each SAW starting from O identifies a distinct endpoint, the expected number of sites connected to O by paths that satisfy the backbend constraint $l_n \leqslant L$ is

$$C_N(L) = p^{N+1} Q_N(L). \tag{12}$$

As $N \to \infty$, $C_N(L)$ vanishes or diverges depending on whether $p < 1/\lambda_0(L)$ or $p > 1/\lambda_0(L)$. Thus the critical concentration for the proliferation of paths with backbends of length *at most* L is

$$p_b(L) = 1/\lambda_0(L) \tag{13}$$

which defines the backbend-constrained percolation threshold. If p_b is known as a function of L, this relation can be implicitly inverted to give $\zeta(p)$. Note that $p_b(\infty) \equiv p_c$ and $p_b(0) \equiv p_d$ as discussed earlier.

We now compute λ_0 for the $(2(L+1)) \times (2(L+1))$ transfer matrix T specified in equation (8). Two rows have vanishing elements, and so T is effectively $2L \times 2L$. Taking $(a_1, \ldots, a_L, b_1, \ldots, b_L)$ as a right eigenvector, the following recurrence equations can be seen to hold:

$$ma_1 + ma_2 + (m-1)b_1 = \lambda a_1 \tag{14a}$$

$$ma_{l+1} + (m-1)b_l = \lambda a_l \qquad \text{for } 2 \leqslant l \leqslant L-1 \tag{14b}$$

$$(m-1)a_l + mb_{l-1} = \lambda b_l \qquad \text{for } 2 \leqslant l \leqslant L \tag{14c}$$

$$(m-1)a_1 = \lambda b_1 \tag{14d}$$

$$(m-1)b_L = \lambda a_L. \tag{14e}$$

Using equations (14a)-(14c), we obtain

$$\begin{pmatrix} a_L \\ b_L \end{pmatrix} = V(\lambda)^{L-2} U(\lambda) \begin{pmatrix} a_1 \\ b_1 \end{pmatrix} \tag{15}$$

where V and U are 2×2 matrices, given by

$$V(\lambda) = \frac{1}{m} \begin{pmatrix} \lambda & 1-m \\ m-1 & (2m-1)/\lambda \end{pmatrix} \qquad U(\lambda) = \frac{1}{m} \begin{pmatrix} \lambda - m & 1-m \\ (m-1)[1-(m/\lambda)] & (2m-1)/\lambda \end{pmatrix}. \tag{16}$$

Let $\Lambda_\pm$ be the eigenvalues of V. Substituting equations (14d) and (14e) into (15), we finally obtain the condition

$$\left(\frac{\Lambda_+(\lambda)}{\Lambda_-(\lambda)}\right)^{L-2} = X_+(\lambda)/X_-(\lambda) \tag{17}$$

where X_+ and X_- are complicated functions of λ. To leading order in $\delta\lambda$ defined as $1-[\lambda/(2m-1)]$, we have

$$X_\pm(\lambda) = \pm 1 - i(2m-1)\left(\frac{2\delta\lambda}{m(m-1)}\right)^{1/2}. \tag{18}$$

For low values of L, the largest eigenvalue $\lambda_0(L)$ and thus $p_b(L)$ can be found by solving equation (17). For the case $m=2$, we have $p_b(0) \equiv p_d = \frac{1}{2}$ and $p_b(\infty) \equiv p_c = \frac{1}{3}$. Further, we find $p_b(1) \simeq 0.414$, $p_b(2) \simeq 0.380$, $p_b(3) \simeq 0.363, \ldots$. In the limit $L \to \infty$, which corresponds to $p_b \to p_c$, we find for general m that

$$\left(\frac{p_b(L) - p_c}{p_c}\right) \approx \frac{m\pi^2}{8(m-1)L^2}. \tag{19}$$

The behaviour of $\zeta(p)$ against p, which follows on inverting the dependence of $p_b(l)$ on l, is shown in figure 2 for $m=2$. Equation (19) implies that the critical exponent μ has the value $\frac{1}{2}$, identical to ν on the Bethe lattice.

A heuristic argument for the equality of μ and ν can be given. Consider an arbitrary lattice, with p close to p_c. The infinite cluster is expected to be well connected on scales greater than ξ. We thus expect that $\zeta \leqslant \text{constant} \times \xi$ and the remaining question is whether ξ can diverge more strongly than ζ as $p \to p_c$.

Imagine applying a renormalisation length scale transformation $\mathcal{R}$ to the system. Since both ξ and ζ are lengths, they are reduced by the same factor $f(>1)$ on each application of $\mathcal{R}$. Let $\mathcal{R}^n$ reduce ζ to order unity; the effective value of p is then close to $p_b(1)$, which is far from the critical region around p_c. Near $p_b(1)$, the correlation length ξ is of order unity as well, which would have been impossible to achieve if ξ had diverged more rapidly than ζ as $p \to p_c$. Therefore, $\zeta \to \text{constant} \times \xi$ as $p \to p_c$ and $\mu \equiv \nu$.

A point to be noted is that, although ξ and ζ diverge in the same fashion as $p \to p_c$, they vanish at differing values of p: $\xi \to 0$ as $p \to 1$, whereas $\zeta \to 0$ as $p \to p_d$. While $p = 1$

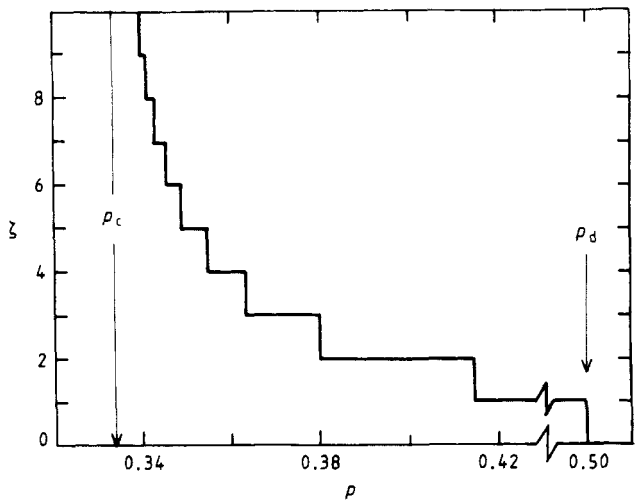

Figure 2. The minimal backbend length ζ as a function of p for a Bethe lattice with $m=2$. The values of p where ζ changes from L to $L+1$ are the thresholds $p_b(L)$. Near p_c (which is $\frac{1}{3}$ on this lattice) ζ diverges as $(p-p_c)^{-1/2}$.

is a stable fixed point of a customary renormalisation group, the directed percolation concentration emerges as a special point (e.g. in the shortest paths problem) only if the evolution of the full probability distribution of lengths is followed under renormalisation [25].

Finally, we turn to the calculation of the chemical lengths of paths satisfying the backbend constraint on the Bethe lattice. Define

$$v_L = \lim_{N\to\infty} (r_N/N) \tag{20}$$

where r_N is the displacement and N is the number of steps on a path with backbend lengths $\leq L$. (The analogous quantity evaluated on a shortest path is the wetting velocity [4].) Consider

$$Z_n(L, h) = \sum_{\{s_n, l_n\}} \prod_{n=1}^{N} \left[T(s_n, l_n; s_{n+1}, l_{n+1}) \exp\left(h \sum_n s_n \right) \right] \tag{21}$$

where h is a field which couples to $r_n = \Sigma_n s_n$, and note that

$$v_L = \lim_{N\to\infty} \frac{1}{N} \frac{\partial}{\partial h} \ln Z_n(L, h). \tag{22}$$

The exponential term in equation (21) can be absorbed into the transfer matrix T and $\lambda_0(L, h)$ can be determined as above. We find that $v_L \approx m/[(m-1)L]$ for large L. Combining this with equation (19) we obtain

$$v_{\zeta(p)} \approx \frac{2}{p} \left(\frac{2m}{m-1} \right)^{1/2} \left(\frac{p-p_c}{p_c} \right)^{1/2} \tag{23}$$

for the chemical lengths of minimal backbend length constrained paths. For the shortest paths, the wetting velocity v_s has been computed before [7]. Both v_s and v_ζ depend on the coordination number $2m$ of the Bethe lattice, but the ratio v_ζ/v_s is $(2/\pi)$, independent of the coordination number!

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